Ways of thinking…what do we mean?

Consider the following situation:

Given that time (t) is measured in years since right now, the function C(t) represents the per capita consumption of soda, in gallons per person, which is currently about 39 gallons per person and is decreasing by about 0.4 gallons per person each year. The function P(t) represents the population of the United States which is currently 327 million people and increasing by 1.5 million people per year. Estimate how the total consumption of soda is changing.

Let's assume that this task or one like it has never been encountered by a student in a Calculus 1 course. However, let's also assume that a student has experienced a Calculus 1 course focused on the major idea of rate of change. What ways of thinking are necessary for a student to successfully respond to this task? Or, put another way, what ways of thinking could a student employ that would lead to effective and productive ways of doing that would result in a successful response to this task?

Ways of Thinking – Part 1: Covarying Quantities

First, a student would have to conceive of the idea of function without necessarily focusing on a specific formula. In this case, we have two functions $C(t)$ and $P(t)$. In the case of $C(t)$, the student would need to conceive of the quantities that covary in tandem. One quantity is the per capita consumption of soda or the number of gallons of soda consumed by an individual person. The other quantity if the number of years that have elapsed since "right now." Therefore, $C(t)$ is a function the demonstrates the relationship between the number of elapsed years and the per capita consumption of soda measured in gallons per person.

Similarly, a student would need to think about the function $P(t)$ as exhibiting the relationship between two more quantities. One quantity is the population of the U.S. measured in "people." The other is, once again, the number of years that have elapsed since "right now." Therefore, $P(t)$ is a function that demonstrates the relationship between the number of elapsed years and the population of the U.S.

Please note that it is unnecessary for students to conceive of these functions as formulas. They may conceive of them as graphs. The graph of $C(t)$ would show that the quantity C decreases as t increases. The graph of $P(t)$ would show that the quantity P increases as t increases. We do have some information about how these quantities are changing currently. However, it is unnecessary to assume a constant rate of change in either case and to therefore create a linear function formula. Rather, it is necessary for students to conceive of both situations (per capita consumption and population) as functions in any (or every) form: graphs, tables, formulas, or just verbal descriptions that we have in this case.

Ways of Thinking – Part 2: Creating a New Function

Next, students need to make sense of the problem situation requiring the relationship between the total consumption (not to be confused with the per capita consumption) of soda and how this new quantity is changing. For students that have powerful and productive ways of thinking about the functions $C(t)$ and $P(t)$, they may be able to create a new function, let's call it $T(t)$, that represents the total amount of soda consumed.

Since $C(t)$ is the amount of soda consumed per person and $P(t)$ represents the number of people t years since "now," the product $C(t)\cdot P(t)$ is the total amount of soda consumed at time t.

Note that, once again, students need not have specific function formulas. Rather, they need to conceive of these functions existing in any from and can use them appropriately. That is, creating a new function that is the product of the given to functions is an important way of thinking for a successful response to this task. That is, $T(t) = C(t)\cdot P(t)$.

Ways of Thinking – Part 3: The Derivative of a Product

The task requests students to think about how the total consumption of soda is changing. That is, how does the function $T(t) = C(t)\cdot P(t)$ change with respect to changes in the elapsed time, t? Given the assumption that the student has studied the idea of derivative in their Calculus 1 course, they may now think about the derivative of $T(t)$ or $T'(t)$. This requires a way of doing…the product rule!

$$T(t) = C(t)\cdot P(t)$$
$$T'(t) = C(t)\cdot P'(t) + C'(t)\cdot P(t)$$

Ways of Thinking – Part 4: Recognizing the Given Quantity Values

Please review the task once again:

Given that time (t) is measured in years since right now, the function C(t) represents the per capita consumption of soda, in gallons per person, which is currently about 39 gallons per person and is decreasing by about 0.4 gallons per person each year. The function P(t) represents the population of the United States which is currently 327 million people and increasing by 1.5 million people per year. Estimate how the total consumption of soda is changing.

Note that we can extract and assign the following quantity values given in the problem statement (where "right now" is $t = 0$):

$$C(0) = 39 \text{ gallons per person}$$
$$C'(0) = -0.4 \text{ gallons per person per year}$$
$$P(t) = 327 \text{ million people}$$
$$P'(t) = 1.5 \text{ million people per year}$$

We now can compute the following using the values above:

$$T(t) = C(t) \cdot P(t)$$

$$T(0) = 39\left[\frac{\text{gallons}}{\text{person}}\right] \cdot 327[\text{million people}]$$

$$T(0) = 12,753[\text{million gallons of soda}]$$

That is, currently there are 12,753,000,000 gallons of soda consumed in the Unites States in the current year.

The task asks, *how the total consumption of soda is changing.*

$$T'(t) = C(t) \cdot P'(t) + C'(t) \cdot P(t)$$

$$T'(0) = 39\left[\frac{\text{gallons}}{\text{person}}\right] \cdot 1.5\ [\text{million people per year}] + (-0.4)\ [\text{gallons per person per year}] \cdot 327[\text{million people}]$$

$$T'(0) = 58.5[\text{million gallons per year}] + (-130.8)[\text{million gallons per year}]$$

$$T'(0) = -72.3[\text{million gallons per year}]$$

That is, the total consumption of soda in the U.S. is currently decreasing at a rate of 72.3 million gallons per year.

Ways of Thinking – Part 5: Analyzing and Interpreting Results

At this point, a thinking student might demonstrate the ability to explain the meaning of the results and also to confirm that the units make sense.

Conclusion

This description is designed to showcase the thinking that a student might engage in to successfully respond to the task. Note that thinking about mathematical meanings often leads to effective and productive ways of doing that then lead to even more mathematical thinking.